REGULARIZING EFFECT OF THE LOWER ORDER TERMS IN SOME NONLINEAR DIRICHLET PROBLEMS

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In this talk Ω is a bounded, open subset of $\mathbb{R}^N$, with $N > 2$, $f(x)$ belongs to $L^m(\Omega)$, with $m \geq 1$, $M(x)$ is a measurable matrix such that

$$\alpha|\xi|^2 \leq M(x)\,\xi\,\xi\,, \qquad |M(x)| \leq \beta\,, \tag{1.1}$$

for almost every x in Ω, and for every ξ in $\mathbb{R}^N$, with $0 < \alpha \leq \beta$.

In [11] is proved (even in a more general setting) the regularizing effect of a polynomial lower order term $g(s) = s|s|^{r-1}$. Indeed for the boundary value problem

$$\begin{cases} -\mathrm{div}\big(M(x)\,\nabla u\big) + u|u|^{r-1} = f \in L^m(\Omega), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases} \tag{1.2}$$

it is possible to prove the existence of weak solutions, even beyond the natural duality pairing; that is: there exists a weak solution $u \in W_0^{1,2}(\Omega) \cap L^{(r-1)m}(\Omega)$, if

$$r' \leq m < \frac{2N}{N+2}, \quad r > 2^*. \tag{1.3}$$

Remark 1.1. Other regularizing effects are studied in [9], [6].

In the previous semilinear problem (1.2), we see that the summability of the solution u increases (recall that Ω is bounded), if the growth of the lower order term increases, even up "infinity growth":

- in [3] is proved the existence of $0 \le u \le M$, with $\mu(\{u(x) = M\}) = 0$, weak solution in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ of
$$-\mathrm{div}\big(M(x)\,\nabla u\big) + \frac{u}{u-M} = f(x) \ge 0.$$
- If $f(x) \ge 0$, in [9] is proved the existence of $0 \le u \in W_0^{1,2}(\Omega)$, weak solution in $W_0^{1,2}(\Omega)$ of
$$-\mathrm{div}\big(M(x)\,\nabla u\big) = \frac{f(x)}{u(x)},$$
even if $f \in L^1(\Omega)$.

In [1] the regularizing effect of the interaction between the coefficient of the zero order term and the datum in some nonlinear elliptic problems is studied.

The simplest example is the linear problem
$$\begin{cases} \displaystyle\int_\Omega M(x)\nabla u\nabla\varphi + \int_\Omega a(x)\,u\,\varphi = \int_\Omega f(x)\varphi, \\ \forall\, \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \end{cases}$$
where $0 \le a(x) \in L^1(\Omega)$. Even if $f(x)$ only belongs to $L^1(\Omega)$, in [1] it is proved that the assumption

$$\text{there exists } Q > 0 \text{ such that } |f(x)| \le Q\,a(x) \tag{1.4}$$

implies the existence of a weak solution u belonging to $W_0^{1,2}(\Omega)$ and such that $|u(x)| \leq Q$:

$$(1.5)\qquad \begin{cases} u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) : \\ \displaystyle\int_\Omega M(x)\nabla u \nabla\varphi + \int_\Omega a(x)\, u\, \varphi = \int_\Omega f(x)\varphi, \\ \forall\, \varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \end{cases}$$

Moreover a simple radial example shows the above boundedness result is sharp.

In [10] it is possible to find an example showing that the bounded solution u of (1.5) is not Hölder-continuous.

The case where Q in (1.4) is not a constant, but a function belonging to some Lebesgue space, is studied in [2].

1.1. Nonlinear principal part.

1.2. Lower order terms having natural growth w.r.t. the gradient (and regularizing effect).

It is well known that the minimization in $W_0^{1,2}(\Omega)$ of simple functionals like

$$J(v) = \frac{1}{2}\int_\Omega (1 + |v|^2)|\nabla v|^2 - \int_\Omega f(x)v(x),$$

leads (at least formally) to the following Euler-Lagrange equation

$$u \in W_0^{1,2}(\Omega) : \ -\mathrm{div}((1 + |u|^2)\nabla u) + u|\nabla u|^2 = f(x).$$

Then the use of $T_1(u)$ as test function gives

$$\int_\Omega |\nabla T_1(u)|^2 + \int_\Omega u\, T_1(u)|\nabla u|^2 \leq \int_\Omega |f(x)|$$

and

$$\int_{\{|u|\leq 1\}} |\nabla u|^2 + \int_{\{1<|u|\}} |\nabla u|^2 \leq \int_\Omega |f(x)|,$$

that is

$$\int_\Omega |\nabla u|^2 \leq \|f\|_{L^1(\Omega)}.$$

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